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STRESS SINGULARITIES AROUND A CRACK IN A COSSERAT PLATE*

N. J. PAGANO
Air Force Materials Laboratory, Dayton, Ohio

and

G. C. SIH
Lehigh University, Bethlehem, Pennsylvania

Abstract—Presented in this paper is the solution of the problem of an infinite plate containing a crack through its entire thickness under uniform bending at infinity. The solution is based upon the recent plate bending theory developed by Green and Naghdi, into which couple-stress is incorporated.

It is shown that the structures of the moment resultant singularities in the couple–stress theory depend upon the elastic constants of the plate. Numerical results are presented which indicate that, for the values of Poisson's ratio considered, the stress intensity factor is always larger than that predicted in the Reissner theory for this problem. It is also observed that an abrupt rise in the magnitudes of the moment resultants near the crack tip occurs in very thin plates as the couple-stress material coefficient increases from zero to a small positive value.

NOTATION

$x_i(i = 1, 2, 3)$	Cartesian coordinate directions
x, y	Cartesian coordinates
ϕ, χ, ψ	scalar functions of x_1 and x_2
∇^2	two-dimensional Laplacian operator
p	applied surface force per unit area
$\alpha_3, \alpha_5, \alpha_6, \alpha_7$	constitutive coefficients
λ^2	a material constant defined by equation (2)
ν	Poisson's ratio
E	modulus of elasticity
D	flexural rigidity
h	plate thickness
$M_{\alpha\beta}(\alpha, \beta = 1, 2)$	components of moment per unit length
$N_{3\alpha}$	components of shear force per unit length
$M_{(\alpha\beta)}$	symmetric part of $M_{\alpha\beta}$
$M_{[\alpha\beta]}$	antisymmetric part of $M_{\alpha\beta}$
$\delta_{\alpha\beta}$	two-dimensional Kronecker delta
$\varepsilon_{\alpha\beta}$	two-dimensional permutation symbol
u_3	deflection of Cosserat surface
δ_α	angles of rotation
s	dummy variable
$A(s), B(s), C(s)$	arbitrary functions of s
$\alpha(s)$	a function of s defined by equation (12)
$2a$	crack length
M	applied bending moment per unit length
$Q(s), f(s), g(s)$	functions of s defined by equations (17) and (21)
$G(x), \phi(t)$	functions defined by equations (22) and (27), respectively
J_0, J_1, I_0, K_0	Bessel functions
R, K, n, A_1	constants defined by equations (35), (47), (48), and (61), respectively

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$h(s)$, $b(s)$	auxiliary functions of s defined by equations (34) and (44), respectively
$\xi, \eta, \Phi(\xi)$	dimensionless variables
$F_1(t, \tau)$, $F(\xi, \eta)$	kernel functions
$U(t)$	unit step function
$\delta(t)$	Dirac delta function
M^*_{11}	non-singular part of M_{11}
$P(\xi, x)$	a function defined by equation (64)
O	order of magnitude symbol
$r, \theta, r_1, \theta_1, r_2, \theta_2$	polar coordinates
$\Phi(1)$, $\mathring{\Phi}(1)$	intensity of singularity in the Green–Naghdi and Reissner theories, respectively
λ_1, L_1	dimensionless material constants
k	constant in the Reissner theory
$\xi, \eta, g(\xi), w(\eta)$	arbitrary variables and functions used in Appendix I
$H(x), \psi(\xi), f(\eta)$	functions defined in Appendix I

INTRODUCTION

THE effects of couple-stress on stress singularities in plate bending have been explored in [1] by considering several fundamental problems involving concentrated forces and moments. The solutions of these problems were achieved through the application of the theory of a Cosserat surface derived in [2] and specialized by Green and Naghdi [3] for the case of an elastic Cosserat plate under small deformations. Briefly, the conclusions reached in [1] were as follows:

(1) The application of Green and Naghdi's couple-stress theory of plate bending alters the structure of the singularities predicted by Reissner's theory [4]. In most cases considered, the maximum intensities of the moment resultants are lower than the corresponding values in Reissner's theory.

(2) The singularities in the couple-stress theory reduce to those in the conventional theory as the couple-stress material constant approaches zero.

Considered in the present paper are the effects of couple-stress on stress singularities in a plate bending problem involving a geometrical discontinuity. Specifically, the problem deals with the stress singularities in a bent plate of infinite extent containing a crack through its entire thickness. The general solution for a semi-infinite plate found in [1] is employed to reduce the solution to a set of dual integral equations. These in turn are transformed into a regular Fredholm integral equation of the second kind. The resulting integral equation is solved numerically by means of an electronic computer.

GOVERNING EQUATIONS

As shown in [3], the governing equations for an elastic, isotropic Cosserat plate under static loading can be conveniently expressed in terms of three potential functions, ϕ, χ, and ψ, as:

$$\begin{aligned} \nabla^2\phi &= -p/\alpha_3 \\ \nabla^2\chi &= \frac{\alpha_3}{D}\phi \\ \left(\nabla^2-\frac{1}{\lambda^2}\right)\psi &= 0 \end{aligned} \tag{1}$$

where p is the intensity of the distributed load normal to the plate surface,

$$\lambda^2 = \alpha_6/\alpha_3 \tag{2}$$

and α_3, α_6, as well as α_5 and α_7 below, are constitutive coefficients. These coefficients are related by

$$\begin{aligned} \alpha_5 &= \nu D \\ \alpha_6+\alpha_7 &= (1-\nu)D \end{aligned} \tag{3}$$

where

$$D = Eh^3/[12(1-\nu^2)] \tag{4}$$

and ν is Poisson's ratio, E is the modulus of elasticity, and h is the plate thickness. The surfaces of the plate are taken to be free from applied couples.

The stress and displacement variables are related to the potential functions [1] by

$$\begin{aligned} M_{(\alpha\beta)} &= \alpha_5\delta_{\alpha\beta}\nabla^2\chi+\frac{(\alpha_6+\alpha_7)}{2}(2\chi_{,\alpha\beta}+\varepsilon_{\alpha\gamma}\psi_{,\gamma\beta}+\varepsilon_{\beta\gamma}\psi_{,\gamma\alpha}) \\ M_{[12]} &= \frac{(\alpha_6-\alpha_7)}{2}\nabla^2\psi \\ N_{3\alpha} &= \alpha_3(\phi_{,\alpha}+\varepsilon_{\alpha\gamma}\psi_{,\gamma}) \\ u_3 &= \phi-\chi \\ \delta_\alpha &= \chi_{,\alpha}+\varepsilon_{\alpha\gamma}\psi_{,\gamma} \end{aligned} \tag{5}$$

where $\delta_{\alpha\beta}$ is the two-dimensional Kronecker delta, $\varepsilon_{\alpha\beta}$ is defined by

$$\begin{aligned} \varepsilon_{11} &= \varepsilon_{22} = 0 \\ \varepsilon_{12} &= -\varepsilon_{21} = 1 \end{aligned} \tag{6}$$

and $x_i (i = 1, 2, 3)$ refers to a right-hand Cartesian coordinate system such that x_1 and x_2 lie in the undeformed Cosserat surface. Also, u_3 is the deflection of the Cosserat surface, δ_α are the angles of rotation shown in Fig. 1, $N_{3\alpha}$ and $M_{\alpha\beta}$ are the shear and moment

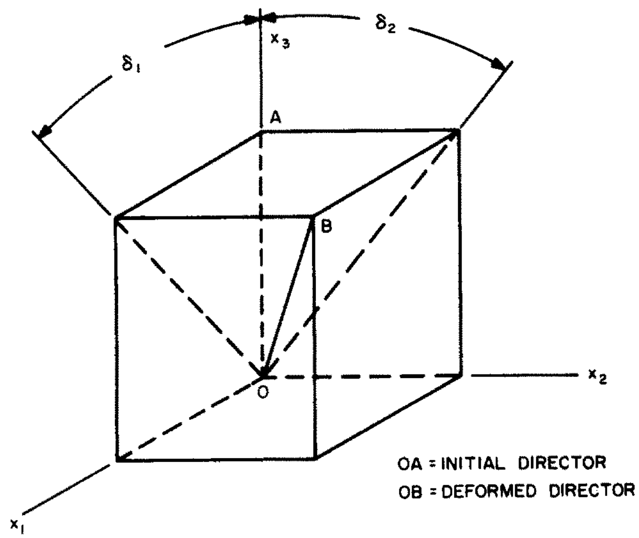

FIG. 1. Definition of δ_α.

resultants (each per unit length), respectively, as displayed in Fig. 2. In equations (5), $M_{\alpha\beta}$ is resolved into its symmetric and antisymmetric parts, given by

$$
\begin{aligned}
M_{(\alpha\beta)} &= M_{(\beta\alpha)} = \tfrac{1}{2}(M_{\alpha\beta}+M_{\beta\alpha}) \\
M_{[\alpha\beta]} &= -M_{[\beta\alpha]} = \tfrac{1}{2}(M_{\alpha\beta}-M_{\beta\alpha}).
\end{aligned}
\tag{7}
$$

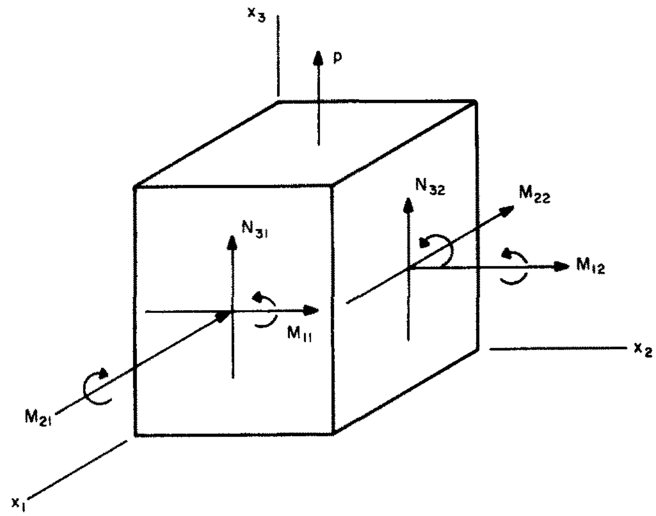

FIG. 2. Definition of $M_{\alpha\beta}$, and $N_{3\alpha}$ and p.

A comma denotes partial differentiation with respect to x_α; Latin indices have the range 1, 2, 3, while Greek indices only take the values 1, 2.

GENERAL SOLUTION FOR A SEMI-INFINITE PLATE

Consider a semi-infinite, elastic, isotropic, homogeneous plate bounded by the planes $x_3 = \pm h/2$ and $x_2 = 0$, where $-\infty < x_1 < \infty$ and $0 \leq x_2 < \infty$. At infinity, it is required that

$$
\left.\begin{aligned}
M_{\alpha\beta} &\to 0 \\
N_{3\alpha} &\to 0
\end{aligned}\right\} \text{ as } (x_\alpha x_\alpha)^{\frac{1}{2}} \to \infty
\tag{8}
$$

Henceforth, the replacements

$$
x_1 \equiv x, \qquad x_2 \equiv y \tag{9}
$$

shall be employed.

In the specific problem discussed later in this paper, the plate will be subjected to mixed (stress and displacement) boundary conditions on the edge $y = 0$ and normal load p is taken to be zero. In this problem, the boundary conditions on $y = 0$ are symmetric with respect to the y-axis. Hence, according to [1], the appropriate solution for the stress and displacement variables is given by

$$
\begin{aligned}
M_{11} &= \frac{1}{\pi}\int_0^\infty \left\langle \left\{ \frac{\alpha_3 A}{2}[2\nu + sy(1-\nu)] - CD(1-\nu)s^2 \right\} e^{-sy} + i(1-\nu)DBs\alpha\, e^{-\alpha y} \right\rangle \cos(sx)\,ds \\
M_{22} &= \frac{1}{\pi}\int_0^\infty \left\langle \left\{ \frac{\alpha_3 A}{2}[2 - sy(1-\nu)] + CD(1-\nu)s^2 \right\} e^{-sy} - i(1-\nu)DBs\alpha\, e^{-\alpha y} \right\rangle \cos(sx)\,ds \\
M_{(12)} &= \frac{(1-\nu)}{2\pi i}\int_0^\infty \{ i[\alpha_3 A(1-sy) + 2CDs^2]\, e^{-sy} + DB(\alpha^2+s^2)\, e^{-\alpha y} \} \sin(sx)\,ds \\
M_{[12]} &= \frac{(\alpha_6-\alpha_7)}{2\pi i\lambda^2}\int_0^\infty B\, e^{-\alpha y} \sin(sx)\,ds \\
N_{31} &= \frac{-\alpha_3}{\pi i}\int_0^\infty (iAs\, e^{-sy} + B\alpha\, e^{-\alpha y}) \sin(sx)\,ds \\
N_{32} &= \frac{\alpha_3}{\pi}\int_0^\infty (-As\, e^{-sy} + iBs\, e^{-\alpha y}) \cos(sx)\,ds
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\delta_1 &= \frac{1}{\pi i}\int_0^\infty \left[i\left(\frac{\alpha_3 Ay}{2D} - Cs \right) e^{-sy} - B\alpha\, e^{-\alpha y} \right] \sin(sx)\,ds \\
\delta_2 &= \frac{1}{\pi}\int_0^\infty \left\{ \left[\frac{\alpha_3 A}{2Ds}(sy-1) - Cs \right] e^{-sy} + iBs\, e^{-\alpha y} \right\} \cos(sx)\,ds \\
u_3 &= \frac{1}{\pi}\int_0^\infty \left[\left(1 + \frac{\alpha_3 y}{2Ds} \right) A - C \right] e^{-sy} \cos(sx)\,ds
\end{aligned}
\tag{11}
$$

where A, B, and C are functions of s and are defined in a particular problem by the boundary conditions on the edge $y = 0$. Also

$$\alpha \equiv \alpha(s) = (s^2 + 1/\lambda^2)^{\frac{1}{2}} \tag{12}$$

BENDING OF A PLATE CONTAINING A CRACK

Consider the problem of an infinite plate $-\infty < x < \infty$, $-\infty < y < \infty$ which contains a crack of length $2a$ parallel to the x axis through the entire thickness and subjected to a bending moment M (per unit length) about the x axis at infinity. The general solution to this problem can be formulated by superposing the solution for the trivial problem of a uniform bending moment and that due to a constant bending moment per unit length applied on the crack boundaries with no loading at infinity. It is the latter problem which will be considered here.

Because of symmetry about the x axis, it can be seen that the functions N_{32} and M_{12} must vanish along the entire x axis, while δ_2 is zero for $|x| > a$, where the origin is taken at the center of the crack. Furthermore, the problem is symmetric with respect to the y axis. Therefore, it is sufficient to consider the region $0 \leq x < \infty$, $0 \leq y < \infty$ under the boundary conditions on $y = 0$,

$$N_{32}(x,0) = M_{12}(x,0) = 0 \qquad \text{for all } x \tag{13}$$

$$M_{22}(x,0) = M, \qquad x < a \tag{14}$$

$$\delta_2(x,0) = 0, \qquad x > a \tag{15}$$

while at infinity, it is required that

$$\begin{matrix} M_{\alpha\beta} \to 0 \\ N_{3\alpha} \to 0 \end{matrix} \qquad \text{as } (x_\alpha x_\alpha)^{\frac{1}{2}} \to \infty$$

Using equations (10), equations (13) are satisfied if

$$\begin{aligned} A(s) &= s^2 Q(s)C(s) \\ B(s) &= -is^2 Q(s)C(s) \end{aligned} \tag{16}$$

where

$$Q(s) = \frac{2(\alpha_6+\alpha_7)}{\alpha_3(1+\nu)+2(\alpha_6+\alpha_7)s^2}. \tag{17}$$

Substituting equations (16) into (10) and (11), using (12), renders

$$\begin{aligned}
M_{11} &= \frac{1}{\pi}\int_0^\infty s^2 C(s)\cos(sx)\left\{\left\langle \frac{\alpha_3}{2}Q(s)[2\nu+(1-\nu)sy]-D(1-\nu)\right\rangle e^{-sy}\right. \\
&\quad \left. +D(1-\nu)sQ(s)\alpha(s)\,e^{-\alpha y}\right\} ds \\
M_{22} &= \frac{1}{\pi}\int_0^\infty s^2 C(s)\cos(sx)\left\{\left\langle \frac{\alpha_3}{2}Q(s)[2-(1-\nu)sy]+D(1-\nu)\right\rangle e^{-sy}\right. \\
&\quad \left. -D(1-\nu)sQ(s)\alpha(s)\,e^{-\alpha y}\right\} ds \\
M_{(12)} &= \frac{(1-\nu)D}{2\pi}\int_0^\infty s^2 C(s)\sin(sx)\left\{\left[\frac{\alpha_3}{D}Q(s)(1-sy)+2\right] e^{-sy}\right. \\
&\quad \left. -Q(s)\left(2s^2+\frac{1}{\lambda^2}\right) e^{-\alpha y}\right\} ds \\
M_{[12]} &= \frac{-(\alpha_6-\alpha_7)}{2\pi\lambda^2}\int_0^\infty s^2 Q(s)C(s)\,e^{-\alpha y}\sin(sx)\,ds \\
N_{31} &= \frac{\alpha_3}{\pi}\int_0^\infty s^2 Q(s)C(s)\sin(sx)[\alpha(s)\,e^{-\alpha y}-se^{-sy}]\,ds \\
N_{32} &= \frac{\alpha_3}{\pi}\int_0^\infty s^3 Q(s)C(s)\cos(sx)(e^{-\alpha y}-e^{-sy})\,ds
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
\delta_1 &= \frac{1}{\pi}\int_0^\infty sC(s)\sin(sx)\left\{sQ(s)\alpha(s)\,e^{-\alpha y}+\left[\frac{\alpha_3 sQ(s)y}{2D}-1\right] e^{-sy}\right\} ds \\
\delta_2 &= \frac{1}{\pi}\int_0^\infty sC(s)\cos(sx)\left\{s^2 Q(s)\,e^{-\alpha y}+\left[\frac{\alpha_3 Q(s)}{2D}(sy-1)-1\right] e^{-sy}\right\} ds \\
u_3 &= \frac{1}{\pi}\int_0^\infty C(s)\,e^{-sy}\cos(sx)\left[\left(1+\frac{\alpha_3 y}{2Ds}\right)s^2 Q(s)-1\right] ds
\end{aligned} \tag{19}$$

Hence, satisfaction of the remaining boundary conditions, equations (14) and (15), requires that

$$\begin{aligned} &\int_0^\infty s^2 C(s) f(s) \cos(sx)\,\mathrm{d}s = \pi M, \qquad x < a \\ &\int_0^\infty sC(s) g(s) \cos(sx)\,\mathrm{d}s = 0, \qquad x > a \end{aligned} \tag{20}$$

where

$$\begin{aligned} f(s) &= \alpha_3 Q(s) + (1-\nu) D[1 - s\alpha(s) Q(s)] \\ g(s) &= Q(s)\left(s^2 - \frac{\alpha_3}{2D}\right) - 1 \end{aligned} \tag{21}$$

Thus, the parameter $C(s)$ is defined by the solution of the dual integral equations (20).

The present object is to transform the dual integral equations (20) into a regular integral equation of a standard form. This is accomplished by first defining a new variable $G(x)$, such that

$$G(x) = \frac{2}{\pi} \int_0^\infty sC(s) g(s) \cos(sx)\,\mathrm{d}s \tag{22}$$

which, with the second of equations (20), implies that

$$sC(s) g(s) = \int_0^a G(x) \cos(sx)\,\mathrm{d}x \tag{23}$$

For convenience, the first of equations (20) is integrated with respect to x. Inserting equation (23) into the resulting equation gives

$$\int_0^\infty \frac{f(s)}{g(s)} \sin(sx)\,\mathrm{d}s \int_0^a G(\eta) \cos(s\eta)\,\mathrm{d}\eta = \pi M x, \qquad x < a \tag{24}$$

Summarizing, $G(x)$ must vanish for $x > a$ and must satisfy equation (24) for $x < a$. The parameter $C(s)$ is then given by equation (23).

Guided by the solution to this problem in the Reissner theory [4] and assuming that couple-stresses will not alter the orders of the stress singularities given in the conventional theory, the solution for $G(x)$ is assumed to be of the form

$$G(x) = \begin{cases} 0 & x > a \\ \displaystyle\int_x^a \frac{\phi(t) t\,\mathrm{d}t}{(t^2 - x^2)^{\frac{1}{2}}}, & x < a \end{cases} \tag{25}$$

where $\phi(t)$ is assumed to be continuous on the interval $[0, a]$. Substituting equation (25) into (23), and applying the Dirichlet formula for interchanging the order of integration [5] gives

$$sC(s) g(s) = \frac{\pi}{2} \int_0^a t\phi(t) J_0(st)\,\mathrm{d}t, \qquad x < a \tag{26}$$

where J_0 is the zero-order Bessel function of the first kind. Similarly, putting equation (25) into (24) yields

$$\int_0^\infty \frac{f(s)}{g(s)} \sin(sx)\, ds \int_0^a t\phi(t) J_0(st)\, dt = 2Mx, \qquad x < a \tag{27}$$

With the aid of equations (17) and (21), one finds

$$\frac{f(s)}{g(s)} = -\frac{(\alpha_6+\alpha_7)}{2\alpha_3}[\alpha_3(3+\nu)+2(\alpha_6+\alpha_7)(s^2-\alpha s)] \tag{28}$$

As shown in [1], the Laurent expansion of $\alpha(s)$ for large s is given by

$$\alpha(s) = s+\frac{1}{2\lambda^2 s}-\frac{1}{8\lambda^4 s^3}+\frac{1}{16\lambda^6 s^5}+O(s^{-7}) \quad \text{as } s \to \infty. \tag{29}$$

Letting

$$f(s)/g(s) = R[h(s)-1] \tag{30}$$

where

$$R = \frac{(\alpha_6+\alpha_7)}{2\alpha_6}[\alpha_6(2+\nu)-\alpha_7] \tag{31}$$

it can be observed that

$$h(s) = O(s^{-2}) \quad \text{as } s \to \infty.$$

Substituting equation (30) into (27) and interchanging the order of integration renders

$$\int_0^a t\phi(t)\, dt \int_0^\infty h(s) J_0(st) \sin(sx)\, ds - \int_0^x \frac{t\phi(t)\, dt}{(x^2-t^2)^{\frac{1}{2}}} = \frac{2Mx}{R}, \qquad x < a \tag{32}$$

where the last term on the left side of the equation follows from an integration with respect to s [7].

Since equation (32) is a form of Abel's equation, as shown in Appendix I, the solution is given by

$$\phi(t) = \frac{2}{\pi}\int_0^t \frac{dx}{(t^2-x^2)^{\frac{1}{2}}}\left[-\frac{2M}{R}+\int_0^a \tau\phi(\tau)\, d\tau \int_0^\infty s h(s) J_0(s\tau)\cos(sx)\, ds\right], \qquad t < a \tag{33}$$

Carrying out the integration with respect to x and introducing the dimensionless variables

$$\xi = t/a, \qquad \eta = \tau/a, \qquad \Phi(\xi) = -R\xi^{\frac{1}{2}}\phi(a\xi)/2M \tag{34}$$

equation (33) becomes

$$\Phi(\xi) = \xi^{\frac{1}{2}}+\int_0^1 \Phi(\eta) F(\xi,\eta)\, d\eta, \qquad \xi < 1 \tag{35}$$

where

$$F(\xi,\eta) = (\xi\eta)^{\frac{1}{2}}\int_0^\infty s h\left(\frac{s}{a}\right) J_0(s\eta) J_0(s\xi)\, ds. \tag{36}$$

The integral in equation (36) is convergent except when $\xi = \eta = 0$, in which case it diverges at the upper limit since the integrand is $O(s^{-1})$ as $s \to \infty$. It is possible to determine the singular part of the integral in closed form and increase its rate of convergence by defining

$$\begin{aligned} b(s) &= h(s) + \frac{(\alpha_6+\alpha_7)^2}{4\alpha_3 R\lambda^2(2\lambda^2 s^2+1)} \\ &= O(s^{-6}) \quad \text{as } s \to \infty \end{aligned} \tag{37}$$

and using the result

$$\int_0^\infty \frac{s}{s^2+n^2} J_0(\xi s) J_0(\eta s)\,\mathrm{d}s = I_0(n\xi)K_0(n\eta), \qquad 0 < \xi \le \eta \tag{38}$$

where I_0 and K_0 are the modified zero-order Bessel functions of the first and second kind, respectively. Putting equation (37) into (36) and using (38) yields

$$\begin{aligned} F(\xi,\eta) = F(\eta,\xi) &= (\xi,\eta)^{\frac{1}{2}}\left[\int_0^\infty sb\left(\frac{s}{a}\right)J_0(s\eta)J_0(s\xi)\,\mathrm{d}s - Ka^2 I_0(na\eta)K_0(na\xi)\right], \\ 0 \le \eta < \xi \le 1 & \end{aligned} \tag{39}$$

where

$$K = \frac{(\alpha_6+\alpha_7)^2}{8\alpha_3 R\lambda^4} = \frac{\alpha_3(\alpha_6+\alpha_7)}{4\alpha_6[\alpha_6(2+\nu)-\alpha_7]} \tag{40}$$

and

$$n^2 = 1/2\lambda^2 = \alpha_3/2\alpha_6. \tag{41}$$

The solution of the problem follows from the solution of the regular Fredholm integral equation of the second kind, equation (35), where the kernel $F(\xi, \eta)$ is defined by equation (39). Once equation (35) is solved, the parameter $C(s)$ may be determined from equations (26) and (34). It is more convenient for subsequent use to integrate equation (26) by parts, with the result that

$$sC(s)g(s) = \frac{\pi a}{2s}\phi(a)J_1(sa) - \frac{\pi}{2s}\int_0^a tJ_1(st)\phi'(t)\,\mathrm{d}t \tag{42}$$

or

$$sC(s)g(s) = \frac{\pi M a}{Rs}\left\{-\Phi(1)J_1(sa) + \int_0^1 \xi J_1(as\xi)\left[\frac{\Phi(\xi)}{\xi^{\frac{1}{2}}}\right]'\mathrm{d}\xi\right\}. \tag{43}$$

The other dependent variables may now be found from equations (18) and (19).

Since $b(s)$ vanishes rapidly for large s according to equation (37), equation (35) can, in general, be easily solved by means of an electronic computer. However, for the case when α_6 and α_7 are both very small, convergence becomes very slow. In fact, for the limiting case when $\alpha_6 = \alpha_7 = 0$, although $\Phi(\xi)$ remains finite, $F(\xi, \eta)$ becomes singular. Thus it is necessary to solve this case analytically. Using equations (28) and (30) with $\alpha_6 = \alpha_7 = 0$ gives

$$h(s) = -2/(1+\nu). \tag{44}$$

Therefore, equation (36) is carried into

$$F(\xi, \eta) = \frac{2(\xi\eta)^{\frac{1}{2}}}{1+\nu}\int_0^\infty sJ_0(s\eta)J_0(s\xi)\,\mathrm{d}s \tag{45}$$

Since

$$\xi\int_0^\infty J_1(\xi s)J_0(\eta s)\,\mathrm{d}s = U(\xi-\eta) \tag{46}$$

where $U(t)$ is the unit step function,

$$U(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} \tag{47}$$

differentiating equation (46) with respect to ξ yields

$$\xi\int_0^\infty sJ_0(\xi s)J_0(\eta s)\,\mathrm{d}s = \delta(\xi-\eta) \tag{48}$$

where $\delta(t)$ is the Dirac delta function. Putting equation (48) into (45) renders

$$F(\xi, \eta) = -\frac{2}{1+\nu}\delta(\xi-\eta). \tag{49}$$

Hence, from equation (35), the closed representation

$$\Phi(\xi) = \frac{(1+\nu)}{(3+\nu)}\xi^{\frac{1}{2}} \tag{50}$$

is obtained.

In order to determine the expressions for the singularities in the shear and moment resultants, the fact that these singularities are only due to the first term on the right-hand side of equation (43) will be demonstrated. For example, consider the influence of the second term on the right side of equation (43) on the moment $M_{11}(x, 0)$. Letting this portion of the moment be represented by $M_{11}^*(x, 0)$, equations (18) and (43) give

$$\begin{aligned} M_{11}^*(x, 0) = \frac{Ma}{R}\int_0^\infty \frac{1}{g(s)}\{Q(s)[\nu\alpha_3+(\alpha_6+\alpha_7)s\alpha] \\ -(\alpha_6+\alpha_7)\}\cos(sx)\,\mathrm{d}s\int_0^1 \xi J_1(as\xi)\left[\frac{\Phi(\xi)}{\xi^{\frac{1}{2}}}\right]'\mathrm{d}\xi \end{aligned} \tag{51}$$

Since the character of this function near the singular point is governed by the behavior at infinity in the s-plane, the following Laurent series are employed:

$$\begin{aligned} \frac{1}{g(s)} &= -\frac{(\alpha_6+\alpha_7)}{\alpha_3}s^2-\frac{(1+\nu)}{2}+O(s^{-2}) \\ Q(s) &= \frac{1}{s^2}-\frac{\alpha_3(1+\nu)}{2(\alpha_6+\alpha_7)s^4}+O(s^{-6}) \end{aligned} \tag{52}$$

as $s \to \infty$, in addition to equation (29) for $\alpha(s)$. Putting these results into equation (51) gives

$$M_{11}^{*}(x, 0) = A_1 \int_0^\infty [1+O(s^{-2})] \cos(sx)\,\mathrm{d}s \int_0^1 \xi J_1(as\xi)\left[\frac{\Phi(\xi)}{\xi^{\frac{1}{2}}}\right]' \mathrm{d}\xi \tag{53}$$

where

$$A_1 = -\frac{Ma(\alpha_6+\alpha_7)(\alpha_7+\nu\alpha_6)}{2R\alpha_6} \tag{54}$$

Using the results

$$\int_0^\infty J_1(as\xi)\cos(sx)\,\mathrm{d}s = \begin{cases} (a\xi)^{-1}, & 0 \le x < a\xi \\ -\dfrac{a\xi}{(x^2-a^2\xi^2)^{\frac{1}{2}}[x+(x^2-a^2\xi^2)^{\frac{1}{2}}]}, & x > a\xi > 0 \end{cases} \tag{55}$$

equation (53) becomes

$$\begin{aligned} M_{11}^{*}(x, 0) = {} & -A_1 a \int_0^{x/a} P(\xi, x)\,\mathrm{d}\xi + \frac{A_1}{a}\int_{x/a}^1 \left[\frac{\Phi(\xi)}{\xi^{\frac{1}{2}}}\right]' \mathrm{d}\xi \\ & + \int_0^1 O(1)\left[\frac{\Phi(\xi)}{\xi^{\frac{1}{2}}}\right]' \mathrm{d}\xi \end{aligned} \tag{56}$$

for the range of $0 \le x \le a$. In the event that $x > a$, the upper limit of the first integral is taken as 1 and the second integral is omitted. In equation (56), the replacement

$$P(\xi, x) = \frac{\xi^2}{(x^2-a^2\xi^2)^{\frac{1}{2}}[x+(x^2-a^2\xi^2)^{\frac{1}{2}}]}\left[\frac{\Phi(\xi)}{\xi^{\frac{1}{2}}}\right]' \tag{57}$$

is employed. Let the first integral in equation (56) be written as

$$\int_0^{x/a} P\,\mathrm{d}\xi = \int_0^{x/a-\varepsilon} P\,\mathrm{d}\xi + \int_{x/a-\varepsilon}^{x/a} P\,\mathrm{d}\xi \tag{58}$$

where ε is a small positive number. The first integrand on the right of equation (58) has no singularities in the interval of integration and the second integral approaches $(\varepsilon)^{\frac{1}{2}}O(1)$ as $\varepsilon \to 0$ and $x \to a$. Therefore, the second term on the right side of equation (43) does not contribute to the singularity in M_{11}. The same result can be illustrated in a similar way for other shear and moment resultants.

In order to determine the closed representations of the stress singularities, it suffices to let

$$C(s) = -\frac{\pi Ma}{Rs^2 g(s)}\Phi(1)J_1(sa) \tag{59}$$

and use the appropriate Laurent expansions in the integrands of equations (18). Recalling equations (29) and (52), and utilizing the Laurent expansion [1],

$$\mathrm{e}^{-\alpha y} = \mathrm{e}^{-sy}\left[1-\frac{y}{2\lambda^2 s}+\frac{y^2}{8\lambda^4 s^2}+\frac{6\lambda^2 y - y^3}{48\lambda^6 s^3}+O(s^{-4})\right] \quad \text{as } s \to \infty \tag{60}$$

equations (18) become

$$
\begin{aligned}
M_{11} &= \frac{M\Phi(1)a(\alpha_6+\alpha_7)(v\alpha_6+\alpha_7)}{2R\alpha_6}\int_0^\infty J_1(sa)\,\mathrm{e}^{-sy}\cos(sx)[1-ys+O(s^{-1})]\,\mathrm{d}s\\
M_{22} &= \frac{M\Phi(1)a(\alpha_6+\alpha_7)}{R\alpha_3}\int_0^\infty J_1(sa)\,\mathrm{e}^{-sy}\cos(sx)\\
&\quad\times\left\{\frac{1}{2\lambda^2}[\alpha_6(2+v)-\alpha_7]+\frac{sy}{2\lambda^2}(v\alpha_6+\alpha_7)+O(s^{-1})\right\}\,\mathrm{d}s\\
M_{(12)} &= \frac{-M\Phi(1)a(\alpha_6+\alpha_7)}{2R\alpha_6}\int_0^\infty J_1(sa)\,\mathrm{e}^{-sy}\sin(sx)[(\alpha_6-\alpha_7)+(v\alpha_6+\alpha_7)sy+O(s^{-1})]\,\mathrm{d}s \qquad (61)\\
M_{[12]} &= \frac{-M\Phi(1)a(\alpha_6-\alpha_7)(\alpha_6+\alpha_7)}{2R\alpha_6}\int_0^\infty J_1(sa)\,\mathrm{e}^{-sy}\sin(sx)[1+O(s^{-1})]\,\mathrm{d}s\\
N_{31} &= \frac{-M\Phi(1)a(\alpha_6+\alpha_7)}{R}\int_0^\infty J_1(sa)\,\mathrm{e}^{-sy}\sin(sx)\left[\frac{y}{2\lambda^2}+O(s^{-1})\right]\mathrm{d}s\\
N_{32} &= \frac{M\Phi(1)a(\alpha_6+\alpha_7)}{R}\int_0^\infty J_1(sa)\,\mathrm{e}^{-sy}\cos(sx)\left[-\frac{y}{2\lambda^2}+O(s^{-1})\right]\mathrm{d}s.
\end{aligned}
$$

Integrating equations (61) (see [7]) and defining r, r_1, r_2, θ, θ_1, and θ_2, as in Fig. 3 yields

$$
\begin{aligned}
M_{11} &= \frac{M\Phi(1)a(v\alpha_6+\alpha_7)}{[\alpha_6(2+v)-\alpha_7](r_1r_2)^{\frac{1}{2}}}\left[-\frac{r}{a}\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)+\frac{ay}{r_1r_2}\sin\frac{3}{2}(\theta_1+\theta_2)\right]+O(1)\\
M_{22} &= \frac{-M\Phi(1)a}{[\alpha_6(2+v)-\alpha_7](r_1r_2)^{\frac{1}{2}}}\\
&\quad\times\left\{[\alpha_6(2+v)-\alpha_7]\frac{r}{a}\cos\left(\theta-\frac{\theta_1\theta_2}{2}\right)+\frac{ay}{r_1r_2}(v\alpha_6+\alpha_7)\sin\frac{3}{2}(\theta_1+\theta_2)\right\}+O(1)\\
M_{(12)} &= \frac{-M\Phi(1)a}{[\alpha_6(2+v)-\alpha_7](r_1r_2)^{\frac{1}{2}}} \qquad (62)\\
&\quad\times\left[(\alpha_6-\alpha_7)\frac{r}{a}\sin\left(\theta-\frac{\theta_1+\theta_2}{2}\right)+\frac{(v\alpha_6+\alpha_7)ay}{r_1r_2}\cos\frac{3}{2}(\theta_1+\theta_2)\right]+O(1)\\
M_{[12]} &= \frac{M\Phi(1)(\alpha_6-\alpha_7)r}{[\alpha_6(2+v)-\alpha_7](r_1r_2)^{\frac{1}{2}}}\sin\left(\theta-\frac{\theta_1+\theta_2}{2}\right)+O(1)\\
N_{31} &= O(1) \qquad N_{32} = O(1)
\end{aligned}
$$

as $r_1r_2 \to 0$.

In the limit as $\alpha_6 \to \alpha_7$, i.e. with couple-stresses absent, equations (62) reduce to

$$
\begin{aligned}
M_{11} &= \frac{M\mathring{\Phi}(1)a}{(r_1r_2)^{\frac{1}{2}}}\left[-\frac{r}{a}\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)+\frac{ay}{r_1r_2}\sin\frac{3}{2}(\theta_1+\theta_2)\right]+O(1)\\
M_{22} &= \frac{-M\mathring{\Phi}(1)a}{(r_1r_2)^{\frac{1}{2}}}\left[\frac{r}{a}\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right)+\frac{ay}{r_1r_2}\sin\frac{3}{2}(\theta_1+\theta_2)\right]+O(1) \qquad (63)\\
M_{12} &= M_{21} = \frac{-M\mathring{\Phi}(1)a^2y}{(r_1r_2)^{\frac{1}{2}}}\cos\frac{3}{2}(\theta_1+\theta_2)+O(1)\\
N_{31} &= O(1) \qquad N_{32} = O(1)
\end{aligned}
$$

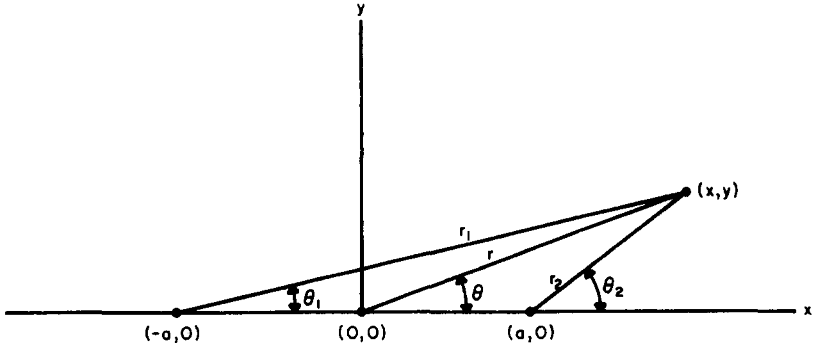

FIG. 3. Polar coordinate systems.

where $\mathring{\Phi}(1)$ is the value of the function $\Phi(1)$ for the case when $\alpha_6 = \alpha_7$. Equations (63) agree with those given by Hartranft and Sih [4] using the Reissner theory of plate bending. It should be noted that the governing equation for $\Phi(\xi)$, equation (35), reduces to the corresponding equation in the Reissner theory [4] in the limit as $\alpha_6 \to \alpha_7$. Hence the stress intensity factor ($\Phi(1)$ in the present notation) is the same in the Reissner theory and in the Green–Naghdi theory as $\alpha_6 \to \alpha_7$.

It is more informative to express the stress distribution in the crack-tip region in terms of polar coordinates r_2, θ_2 with respect to an origin at the point $(a, 0)$. To this end, it is observed that $r_1 \to 2a$, $r \to a$, $\theta_1 \to 0$, $\theta \to 0$ as $r_2 \to 0$. Thus equations (62) may be written as

$$\begin{aligned}
M_{11} &= \frac{M\Phi(1)a^{\frac{1}{2}}(\nu\alpha_6+\alpha_7)}{[\alpha_6(2+\nu)-\alpha_7](2r_2)^{\frac{1}{2}}}\left[-\cos\frac{\theta_2}{2}+\tfrac{1}{2}\sin\theta_2\sin\tfrac{3}{2}\theta_2\right]+O(1)\\
M_{22} &= \frac{-M\Phi(1)a^{\frac{1}{2}}}{[\alpha_6(2+\nu)-\alpha_7](2r_2)^{\frac{1}{2}}}\left\{[\alpha_6(2+\nu)-\alpha_7]\cos\frac{\theta_2}{2}+\frac{(\nu\alpha_6+\alpha_7)}{2}\sin\theta_2\sin\tfrac{3}{2}\theta_2\right\}+O(1)\\
M_{(12)} &= \frac{-M\Phi(1)a^{\frac{1}{2}}}{[\alpha_6(2+\nu)-\alpha_7](2r_2)^{\frac{1}{2}}}\left[-(\alpha_6-\alpha_7)\sin\frac{\theta_2}{2}+\frac{(\nu\alpha_6+\alpha_7)}{2}\sin\theta_2\cos\tfrac{3}{2}\theta_2\right]+O(1)\\
M_{[12]} &= \frac{-M\Phi(1)(\alpha_6-\alpha_7)a^{\frac{1}{2}}}{[\alpha_6(2+\nu)-\alpha_7](2r_2)^{\frac{1}{2}}}\sin\frac{\theta_2}{2}+O(1)\\
N_{31} &= O(1) \qquad N_{32} = O(1)
\end{aligned} \tag{64}$$

as $r_2 \to 0$.

For completeness, and to check the singular functions of equations (64), these functions were substituted into the governing equations in the present theory and the deflection u_3 and angular displacements δ_α in the crack-tip region were computed. This work is presented in Appendix II.

While the orders of the singularities in the moment resultants in the present theory and the Reissner theory are the same, their functional relations in x and y are different. In contrast to equations (63), the angular distribution of some of the singular functions in equations (62) depends upon the elastic moduli. It is also interesting to make some qualitative observations regarding the effects of couple-stress on the material behavior in the region ahead of the crack, i.e. for $\theta_2 = 0$, the direction in which the crack would be expected to run when it reaches its critical length. Along this line, the Reissner theory predicts that the ratio M_{11}/M_{22} is equal to unity while $M_{12} = M_{21} = 0$. In the couple-stress theory, along $\theta_2 = 0$, one finds that

$$\frac{M_{11}}{M_{22}} = \frac{\nu\alpha_6+\alpha_7}{\alpha_6(2+\nu)-\alpha_7}. \tag{65}$$

However, since the strain energy function must be positive definite, it is necessary to require that [1]

$$\begin{gathered}\alpha_6 \geq 0 \\ (\alpha_6-\alpha_7) \geq 0.\end{gathered} \tag{66}$$

Thus, the corresponding ratio in the couple-stress problem must satisfy

$$-\frac{1-\nu}{3+\nu} \leq \frac{M_{11}}{M_{22}} \leq 1. \tag{67}$$

It appears that couple-stresses can effect an appreciable revision in the hydrostatic component of stress, as compared to the corresponding component predicted in the Reissner theory. Thus the material behavior, e.g. its yielding characteristics, may be considerably influenced by couple-stresses.

In order to gain further insight regarding the influence of couple-stress in plate bending, equation (35) was solved numerically. The curves of Figs. 4 through 10 represent certain features of the solution. In these curves, three values of Poisson's ratio are considered. In addition, the following dimensionless constants are introduced

$$\begin{aligned}\lambda_1^2 &= \lambda^2/a^2 = \alpha_6/a^2\alpha_3 \\ L_1^2 &= L^2/a^2 = (\alpha_6-\alpha_7)/a^2\alpha_3.\end{aligned} \tag{68}$$

If the parameter λ is identified with the constant k in the Reissner theory, as suggested in [1], i.e.,

$$\lambda^2 = k^2 \tag{69}$$

then λ_1^2 may be expressed as

$$\lambda_1^2 = k^2/a^2 = h^2/10a^2. \tag{70}$$

Thus, λ_1 is a measure of the ratio of plate thickness to crack length. The constant L is a material length parameter or couple-stress coefficient.

Figure 4 represents the stress intensity factor $\Phi(1)$ as a function of λ_1 for various values of L_1 with $\nu = 0$. The minimum value of λ_1 for each curve is limited by the condition $\alpha_6+\alpha_7 \geq 0$. The curve for $L_1 = 0$, which corresponds to the Reissner solution, appears to represent a lower bound for $\Phi(1)$. A finite jump discontinuity exists in the neighborhood of $L_1 = 0$. This behavior was also observed in [6]. Thus, in a plate having a very low h/a ratio, the effects of couple-stresses can be very severe. It is noted that all of the curves in Fig. 4 tend to merge and asymptotically approach a limiting value $\Phi(1) = 1$ for very large values of λ_1. The ordinate of each curve also attains a maximum value of unity at the point where $\alpha_7 = -\alpha_6$ or $L_1 = (2)^{\frac{1}{2}}\lambda_1$ provided $\lambda_1 > 0$. This point physically represents a plate of vanishing flexural rigidity D. The value $\Phi(1) = 1$ for $L_1 = (2)^{\frac{1}{2}}\lambda_1(\lambda_1 > 0)$ can be established analytically by considering equations (28), (30), (35), and (36) with $\alpha_7 = -\alpha_6 \neq 0$.

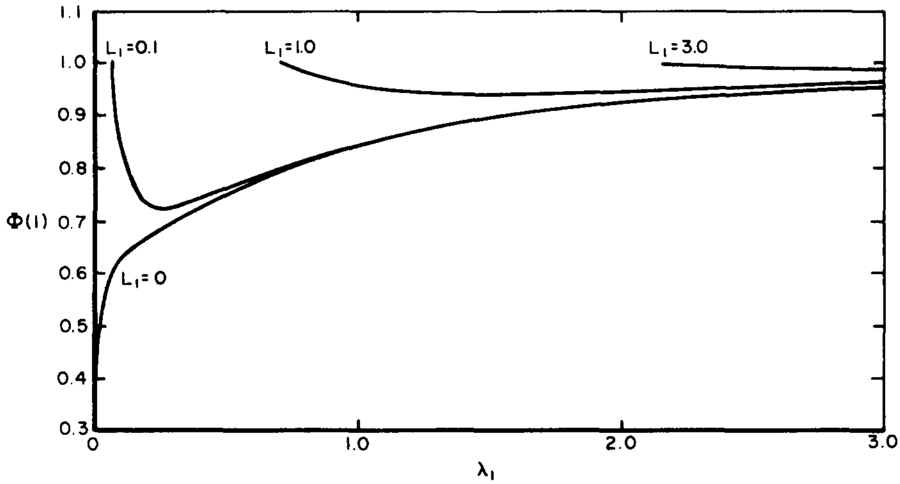

FIG. 4. Intensity of singularity vs. λ_1, $\nu = 0$.

The curves of Figs. 5 and 6 illustrate the same type of behavior as discussed above for different values of Poisson's ratio, and therefore require no additional comment. It is observed, however, that the curve representing the Reissner solution for $\nu = 0{\cdot}3$ is in agreement with that given by Hartranft and Sih [4]. Curves showing the relationship between $\Phi(1)$ and L_1 are plotted in Figs. 7, 8, and 9 for various values of λ_1 and ν. These clearly show that the Reissner solution ($L_1 = 0$) represents the lower bound for $\Phi(1)$ in each case considered. This result is in qualitative agreement with the results presented in [6] for an analogous plane couple-stress elasticity problem.

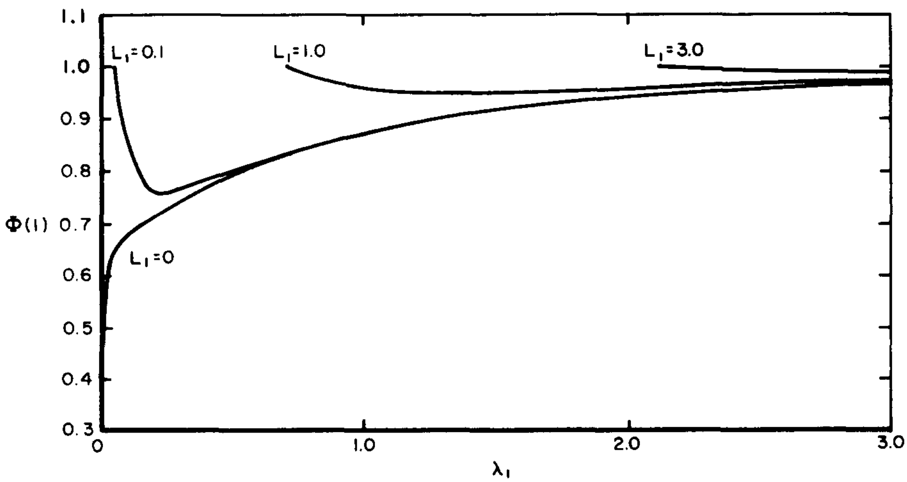

FIG. 5. Intensity of singularity vs. λ_1, $\nu = 0{\cdot}3$.

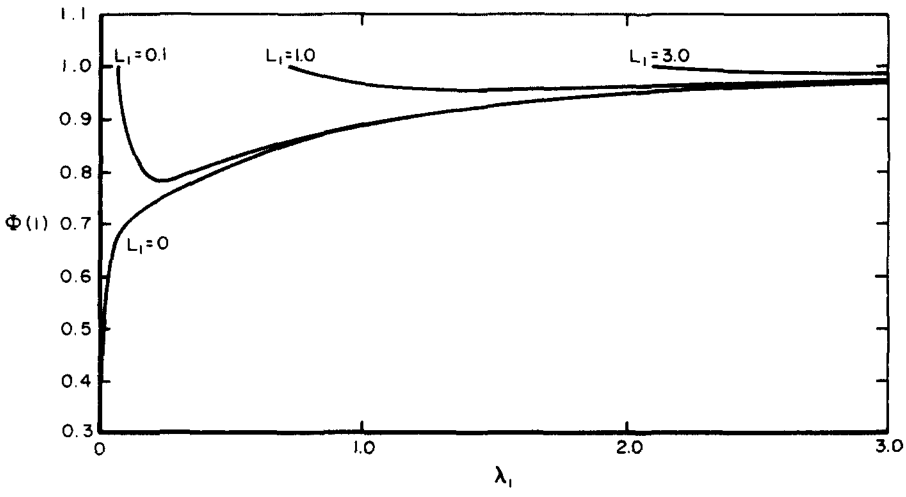

FIG. 6. Intensity of singularity vs. λ_1, $\nu = 0{\cdot}5$.

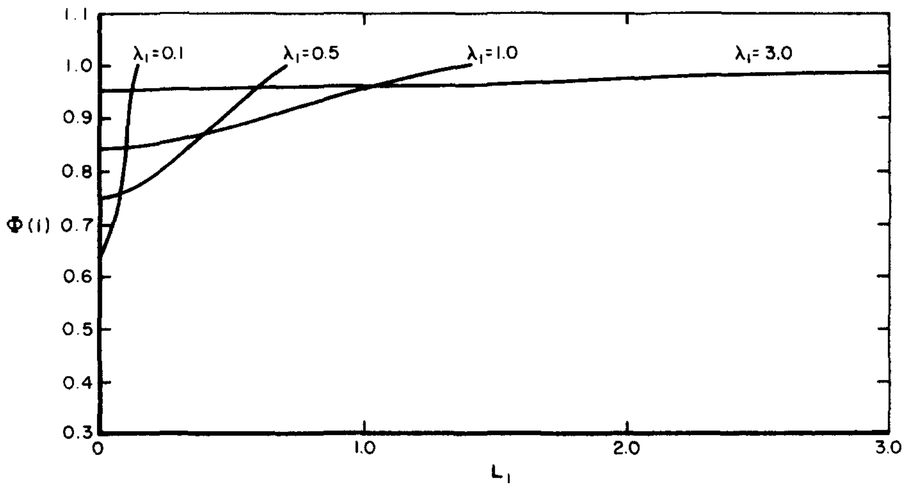

FIG. 7. Intensity of singularity vs. L_1, $\nu = 0$.

Since the preceding curves cannot be expected to be valid for large values of λ_1 in the present thin plate theory, blown-up curves for $L_1 = 0{\cdot}1$ and small λ_1 are drawn in Fig. 10.

CONCLUSIONS

Some insight as to the effect of couple-stress in plate bending, at least on a qualitative basis, has been achieved through the investigation of the crack-tip stress field in a Cosserat plate. Since the ratio M_{11}/M_{22} ahead of the crack may depart considerably from the value

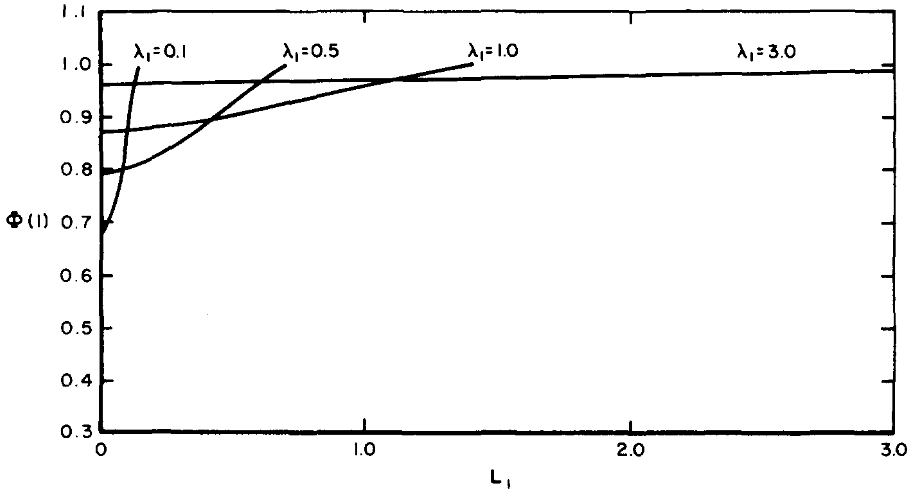

FIG. 8. Intensity of singularity vs. L_1, $\nu = 0{\cdot}3$.

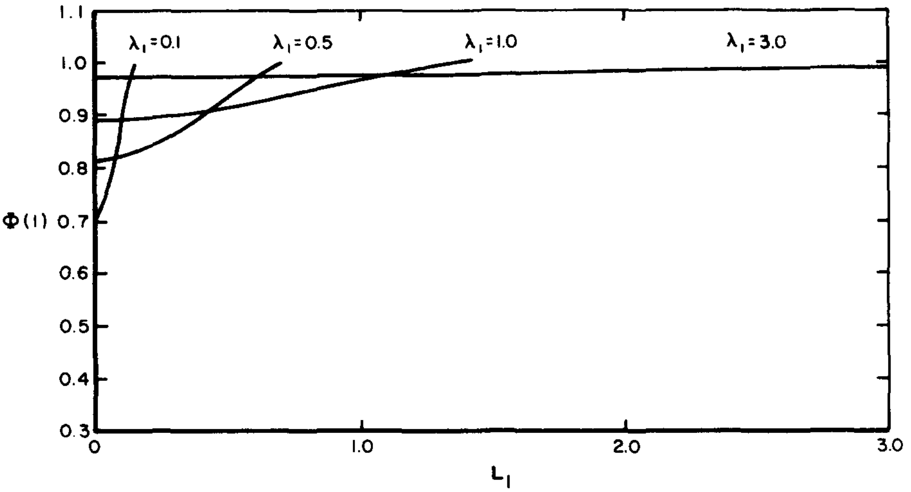

FIG. 9. Intensity of singularity vs. L_1, $\nu = 0{\cdot}5$.

of unity predicted in the Reissner theory, the extent of inelastic deformation in this region may be appreciably influenced.

The Reissner solution for the stress intensity factor is the lower bound for all cases considered, although this is expected to be true in general. This result is in qualitative agreement with the results presented in [6] for a corresponding plane couple-stress elasticity problem.

It is interesting to note that, as in [1], the singularities in the moment resultants in the present theory reduce to those given in the Reissner theory as $L \to 0$. In [6], the analogous limit depends discontinuously on the material length parameter. This fact is evidently explained by consideration of the boundary conditions. In plate bending problems, using the present formulation, the number of boundary conditions remains unaltered as L approaches zero. The reduction of couple-stress elasticity to classical elasticity, however, is accompanied by a reduction in the number of boundary conditions (as well as a lowering of the order of the system of governing equations), which appears to represent the source of the discontinuity.

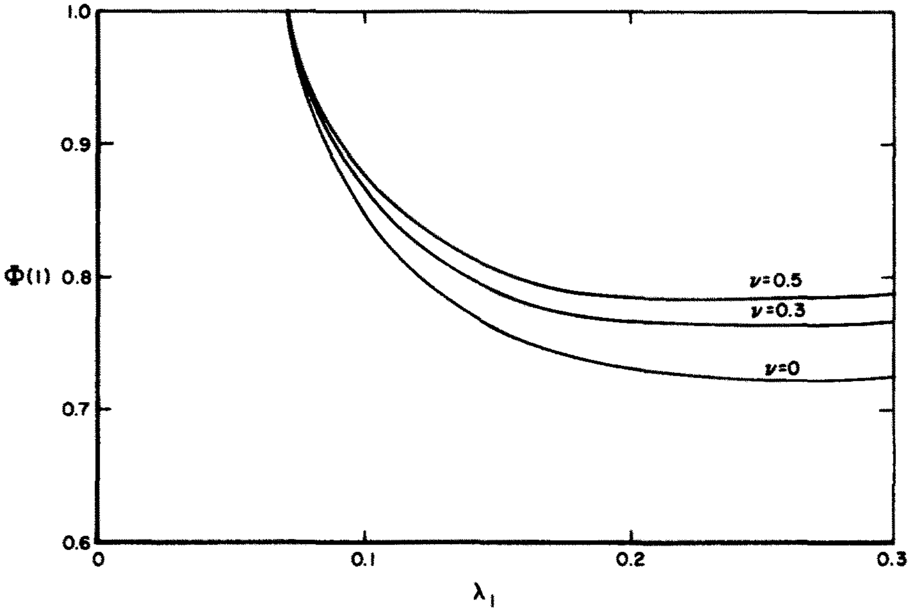

FIG. 10. Intensity of singularity vs. L_1, $\nu = 0{\cdot}1$.

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APPENDIX I

Solution of Equation (32)

The general form of Abel's equation is given by

$$\int_0^\eta \frac{g(\xi)\,\mathrm{d}\xi}{(\eta-\xi)^\alpha} = w(\eta), \qquad \eta \geq 0, \qquad 0 < \alpha < 1 \tag{71}$$

for which the unique solution is

$$g(\xi) = \frac{\sin\alpha\pi}{\pi}\left[\int_0^\xi \frac{w'(\eta)\,\mathrm{d}\eta}{(\xi-\eta)^{1-\alpha}} + \frac{w(0)}{\xi^{1-\alpha}}\right] \tag{72}$$

Equation (32) may be written as

$$\int_0^x \frac{t\phi(t)\,\mathrm{d}t}{(x^2-t^2)^{\frac{1}{2}}} = H(x), \qquad 0 \leq x < a \tag{73}$$

where

$$H(x) = \frac{-2Mx}{R} + \int_0^a t\phi(t)\,\mathrm{d}t \int_0^\infty h(s)J_0(st)\sin(sx)\,\mathrm{d}s \tag{74}$$

$$H(0) = 0.$$

In equation (73), let

$$x^2 = \eta, \qquad t^2 = \xi$$

so that

$$2t\,\mathrm{d}t = \mathrm{d}\xi, \qquad \int_0^x \mathrm{d}t = \int_0^{x^2=\eta} \mathrm{d}\xi$$

which transform equation (73) into

$$\int_0^\eta \frac{\psi(\xi)\,\mathrm{d}\xi}{(\eta-\xi)^{\frac{1}{2}}} = f(\eta), \qquad 0 \leq \eta < a^2 \tag{75}$$

where

$$\begin{aligned} \psi(\xi) &= \phi(\xi^{\frac{1}{2}}) \\ f(\eta) &= 2H(\eta^{\frac{1}{2}}) \end{aligned} \tag{76}$$

Thus, the solution of equation (75) is expressed by

$$\psi(\xi) = \frac{1}{\pi}\int_0^\xi \frac{f'(\eta)\,\mathrm{d}\eta}{(\xi-\eta)^{\frac{1}{2}}}, \qquad 0 \leq \xi < a^2. \tag{77}$$

Letting

$$\xi = t^2, \qquad \eta = x^2$$

and using

$$f'(\eta)\,\mathrm{d}\eta = 2H'(x)\,\mathrm{d}x$$

equation (77) becomes

$$\psi(t^2) = \phi(t) = \frac{2}{\pi}\int_0^t \frac{H'(x)\,\mathrm{d}x}{(t^2-x^2)^{\frac{1}{2}}}, \qquad 0 \geq t < a. \tag{78}$$

Putting equation (74) into (78) leads directly to equation (33).

APPENDIX II

Complete Solution in the Crack Tip Region

In order to determine the complete solution in the crack tip region, it is necessary to write the pertinent governing equations in terms of polar coordinates. For sake of familiarity, let the polar coordinates r_2, θ_2 in Fig. 3 be replaced by r, θ. By standard techniques, the following equations may be written

$$\begin{aligned} M_{rr,r}+\frac{1}{r}M_{r\theta,\theta}+\frac{1}{r}(M_{rr}-M_{\theta\theta}) &= N_{3r} \\ M_{\theta r,r}+\frac{1}{r}M_{\theta\theta,\theta}+\frac{1}{r}(M_{r\theta}+M_{\theta r}) &= N_{3\theta} \end{aligned} \tag{79}$$

$$\begin{aligned} \kappa_{rr} = \delta_{r,r}, \qquad \kappa_{\theta\theta} &= \frac{1}{r}\delta_r+\frac{1}{r}\delta_{\theta,\theta} \\ \kappa_{r\theta} = \frac{1}{r}(\delta_{r,\theta}-\delta_\theta), \qquad \kappa_{\theta r} &= \delta_{\theta,r} \end{aligned} \tag{80}$$

$$\begin{aligned} M_{rr} &= \alpha_5(\kappa_{rr}+\kappa_{\theta\theta})+(\alpha_6+\alpha_7)\kappa_{rr} \\ M_{\theta\theta} &= \alpha_5(\kappa_{rr}+\kappa_{\theta\theta})+(\alpha_6+\alpha_7)\kappa_{\theta\theta} \\ M_{r\theta}+M_{\theta r} &= (\alpha_6+\alpha_7)(\kappa_{r\theta}+\kappa_{\theta r}) \\ M_{r\theta}-M_{\theta r} &= (\alpha_6-\alpha_7)(\kappa_{r\theta}-\kappa_{\theta r}) \end{aligned} \tag{81}$$

$$N_{3r,r}+\frac{1}{r}N_{3r}+\frac{1}{r}N_{3\theta,\theta} = 0 \tag{82}$$

$$\begin{aligned} N_{3r} &= \alpha_3(\delta_r+u_{3,r}) \\ N_{3\theta} &= \alpha_3\left(\delta_\theta+\frac{1}{r}u_{3,\theta}\right) \end{aligned} \tag{83}$$

where the dependent variables in the polar coordinate system can be defined by replacing 1 by r and 2 by θ in Figs. 1 and 2.

By use of the relations

$$\begin{aligned} \sin A \cos B &= \tfrac{1}{2}[\sin(A+B)+\sin(A-B)] \\ \sin A \sin B &= \tfrac{1}{2}[\cos(A-B)-\cos(A+B)] \end{aligned} \tag{84}$$

the functions in equations (64) become

$$
\begin{aligned}
M_{11} &= \frac{-(\nu\alpha_6+\alpha_7)C_1}{4r^{\frac{1}{2}}}\left(3\cos\frac{\theta}{2}+\cos\frac{5}{2}\theta\right)+O(1)\\
M_{22} &= \frac{-C_1}{4r^{\frac{1}{2}}}\left\{[\alpha_6(8+5\nu)-3\alpha_7]\cos\frac{\theta}{2}-(\nu\alpha_6+\alpha_7)\cos\frac{5}{2}\theta\right\}+O(1)\\
M_{(12)} &= \frac{-C_1}{4r^{\frac{1}{2}}}\left\{[-(4+\nu)\alpha_6+3\alpha_7]\sin\frac{\theta}{2}+(\nu\alpha_6+\alpha_7)\sin\frac{5}{2}\theta\right\}+O(1)\\
M_{[12]} &= \frac{-C_1(\alpha_6-\alpha_7)}{r^{\frac{1}{2}}}\sin\frac{\theta}{2}+O(1)\\
N_{31} &= O(1), \qquad N_{32} = O(1)
\end{aligned}
\tag{85}
$$

as $r \to 0$, where

$$
C_1 = \frac{M\Phi(1)}{[\alpha_6(2+\nu)-\alpha_7]}\left(\frac{a}{2}\right)^{\frac{1}{2}}. \tag{86}
$$

Transforming equations (85) into polar coordinates and use of (84) renders

$$
\begin{aligned}
M_{rr} &= \frac{-C_1}{4r^{\frac{1}{2}}}\left\{[-(4+\nu)\alpha_6+3\alpha_7]\cos\frac{3}{2}\theta+[(4+5\nu)\alpha_6+\alpha_7]\cos\frac{\theta}{2}\right\}+O(1)\\
M_{\theta\theta} &= \frac{-C_1}{4r^{\frac{1}{2}}}\left\{[(4+\nu)\alpha_6-3\alpha_7]\cos\frac{3}{2}\theta+[(4+3\nu)\alpha_6-\alpha_7]\cos\frac{\theta}{2}\right\}+O(1)\\
M_{r\theta} &= \frac{-C_1[\alpha_6(4+\nu)-3\alpha_7]}{4r^{\frac{1}{2}}}\left(\sin\frac{3}{2}\theta+\sin\frac{\theta}{2}\right)+O(1)\\
M_{\theta r} &= \frac{-C_1}{4r^{\frac{1}{2}}}\left\{[(4+\nu)\alpha_6-3\alpha_7]\sin\frac{3}{2}\theta-[(4-\nu)\alpha_6-5\alpha_7]\sin\frac{\theta}{2}\right\}+O(1)\\
N_{3r} &= O(1), \qquad N_{3\theta} = O(1).
\end{aligned}
\tag{87}
$$

Direct substitution shows that the singular parts of the moment resultants satisfy the equations of equilibrium (79) with $N_{3r} = N_{3\theta} = 0$ since there are no singularities in the shear resultants. Using the third and fourth of each set of equations (80), (81), and (87), one can show that

$$
\delta_{\theta,r} = -\frac{1}{r^{\frac{1}{2}}}\left(C_2\sin\frac{3}{2}\theta+C_3\sin\frac{\theta}{2}\right) \tag{88}
$$

or

$$
\delta_\theta = -2r^{\frac{1}{2}}\left(C_2\sin\frac{3}{2}\theta+C_3\sin\frac{\theta}{2}\right)+f'(\theta) \tag{89}
$$

where $f'(\theta)$ is an arbitrary function of θ, and also

$$
\delta_{r,\theta} = -r^{\frac{1}{2}}\left(3C_2\sin\frac{3}{2}\theta+C_4\sin\frac{\theta}{2}\right)+f'(\theta). \tag{90}
$$

In equations (88)–(90), the following contractions are employed:

$$C_2 = -\frac{[\alpha_6(4+\nu)-3\alpha_7]C_1}{4(\alpha_6+\alpha_7)}$$
$$C_3 = -\frac{[\alpha_6(4-\nu)+3\alpha_7]C_1}{4(\alpha_6+\alpha_7)} \tag{91}$$
$$C_4 = -\frac{[\alpha_6(4-3\nu)+\alpha_7]C_1}{4(\alpha_6+\alpha_7)}.$$

Integration of equation (90) yields

$$\delta_r = 2r^{\frac{1}{2}}\left(C_2\cos\frac{3}{2}\theta+C_4\cos\frac{\theta}{2}\right)+f(\theta)+g(r) \tag{92}$$

where $g(r)$ is an arbitrary function of r. Equations (89) and (92) are now put into the first two of each set of equations (80) and (81). In conjunction with the first two of equations (87), this yields

$$\begin{aligned} rg'(r)+\nu[f(\theta)+f''(\theta)+g(r)] &= 0 \\ \nu rg'(r)+f(\theta)+f''(\theta)+g(r) &= 0. \end{aligned} \tag{93}$$

The general solution of equations (93) is given by

$$\begin{aligned} g(r) &= F \\ f(\theta) &= -F+H\sin\theta+S\cos\theta \end{aligned} \tag{94}$$

where F, H, and S are arbitrary constants. Since $\delta_\theta(r, 0) = 0$, equations (89) and (94) require that $H = 0$. Therefore, the angular displacements are

$$\begin{aligned} \delta_r &= 2r^{\frac{1}{2}}\left(C_2\cos\frac{3}{2}\theta+C_4\cos\frac{\theta}{2}\right)+S\cos\theta \\ \delta_\theta &= -2r^{\frac{1}{2}}\left(C_2\sin\frac{3}{2}\theta+C_3\sin\frac{\theta}{2}\right)-S\sin\theta. \end{aligned} \tag{95}$$

The constant S cannot be determined unless the solution is extended beyond the region near the crack tip.

The governing equation for u_3 is found by substituting equations (83) into (82), with the result that

$$u_{3,rr}+\frac{1}{r}u_{3,r}+\frac{1}{r^2}u_{3,\theta\theta} = -\left(\delta_{r,r}+\frac{1}{r}\delta r+\frac{1}{r}\delta_{\theta,\theta}\right). \tag{96}$$

The boundary conditions for equation (96) are given by substituting

$$N_{3\theta}(r, 0) = N_{3\theta}(r, \pi) = 0 \tag{97}$$

into equations (83). Inserting equations (95) into (96) renders

$$\nabla^2 u_3 = (3C_4-C_3)r^{-\frac{1}{2}}\cos\frac{\theta}{2}. \tag{98}$$

The solution of equation (98) that satisfies equations (97) is given by

$$u_3 = A+\frac{r^{\frac{3}{2}}}{6}\left[(9C_3-8C_2-3C_4)\cos\frac{3}{2}\theta+(3C_3-9C_4)\cos\frac{\theta}{2}\right] \tag{99}$$

where A is an arbitrary constant.

The shear resultants in the crack tip region are given by

$$\begin{aligned} N_{3\theta} &= \alpha_3\left[\frac{3r^{\frac{1}{2}}}{4}(C_4-3C_3)\left(\sin\frac{\theta}{2}+\sin\frac{3}{2}\theta\right)-S\sin\theta\right] \\ N_{3r} &= \alpha_3\left[(3C_3-C_4)\frac{r^{\frac{1}{2}}}{4}\left(\cos\frac{\theta}{2}+3\cos\frac{3}{2}\theta\right)+S\cos\theta\right] \end{aligned} \tag{100}$$

which follow from equations (83), (95), and (99).

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Абстракт—В настоящей работе представляется решение задачи бесконечной пластинки со сквозной трещиной, подверженной постоянному изгибу в бесконечности. Решение основано на последедней теории изгиба пластинки, разработаной Грином и Нагхди, которая содержит моментное напряжение. Оказывается, что структура суммарных моментных сингулярностей зависит от упругих постоянных пластинки. Представляются численные результаты, которые указывают на то, что когда прининается во внимение значение отношения Пуассона, тогда коэффициент интенсивности напряжения является всегда большим, чум это следует из теории Рейсснера для этой задачи. Наблюдается Также, что внезапный рост величин суммарных моментов близи конца трещины происходит в очень тонких пластинках, когда коэффициент моментного напряжения повышается от нуля до малой положителрной величины.